Free Boundary Problems

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Thanks to Leoni

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Introduction and Context

Introduction

A free boundary problem is a differential equation where the solution is not only unknown, but also part of the region in which it is defined is unknown and must be determined as part of the solution.
We will focus on a particular type of free boundary problems, the obstacle problems. To look into that, we need Calculus of variations, which specifically looks at how to optimize certain functionals given certain constraints

Definition

Def: A functional is a function G which takes in a function f from a given domain, and outputs a scalar. An example of a function could be:

$$G : C^1[0,1] \to \mathbb{R}, G(h) = \int_0^1 h'(x)^2 dx$$

If we restrict the domain of G to functions h such that $h(0) = 0, h(1) = 1$, when is $G(h)$ minimal?

The Euler-Lagrange Equation

Theorem

Suppose that $J(y) = \int_a^b F(x, y, y')dx$ *where* $J : C^{1,pw}[a, b] \to \mathbb{R}$. *Let* $y \in D \subset C^{1,pw}([a, b])$ *be a local minimizer of the functional* J, *and assume that the Lagrange function* $F : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and continuously partially differentiable with respect to the last two variables. Then the following are true:*

$$\frac{\partial F}{\partial y'} \in C^{1,pw}([a, b]) \subset C^1([a, b])$$

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'} \quad \text{piecewise on } [a, b]$$

The Euler-Lagrange Equation

The proof of the Euler-Lagrange equation uses the idea of directional derivatives in the space of functions:

$$J'_h(y) = \lim_{t \to 0} \frac{J(y+th) - J(y)}{t}$$

If y is an global minimizer, then we know that $J'_h(y) = 0$. However, working it out by integration by parts yields that if $J'_h(y) = 0$ then:

$$J'_h(y) = \int_a^b (F_y - \frac{d}{dx} F_{y'})h = 0$$

and since y is a global minimizer, this is true for all $h \in C^{1,pw}[a,b]$, which implies again by integration by parts that $F_y - \frac{d}{dx} F'_y$ is identically zero.

Obstacles

But now, we introduce obstacles into the equations. Suppose we let $g : [a, b] \to \mathbb{R}$ be C^2 and we restrict the domain of J to be functions h such that $h \geq g, h \in C^{1,pw}[a, b]$ only? Adapting the proof of the Euler-Lagrange, Equation yields:

$$\lim_{t \to 0^+} \frac{J(y + th) - J(y)}{t} \geq 0$$

for all $h \in C^{1,pw}[a, b], h \geq 0$. Applying a similar technique from before yields:

$$J_h'(y) = \int_a^b (F_y - \frac{d}{dx} F_{y'})h \geq 0$$

for all h positive, which means that $\int F_y - F_{y'}$ is decreasing over $[a, b]$. This is the Euler Lagrange Inequality, and under appropriate regularity conditions,

$$\frac{d}{dx} F_{y'} - F_y \geq 0$$

An introductory Example

Suppose that $F(x, y, y') = |y'|^2$, and we want to find y, an optimal solution over all $y \geq g$, g is C^2. Then:

$$y > g \implies y'' = 0$$

$$y = g \implies y'' \leq 0$$

Furthermore, since g'' is continuous, suppose $g'' \leq 0, y > g$ on $[r, s]$, $g(r) = y(r), g(s) = r(s)$. This turns out not to be possible by convexity.
Now, if $y > g$ on $[a, b]$ it's possible to argue why if y is optimal then y must be completely linear on $[a, b]$, not just piecewise linear.

C^2 Lower and Upper solutions

Existence of Solutions to ODEs

We consider the problem

$$u'' = f(t, u, u')$$

$$a_1 u(a) - a_2 u'(a) = A$$

$$b_1 u(b) + b_2 u'(b) = B$$

for some constants A, B, a_1, b_1, and some positive a_2, b_2.

Definition

Let $u : [a, b] \to \mathbb{R}$ continuous, $t_0 \in (a, b)$. The lower right Dini derivative is

$$D_+ u(t_0) := \liminf_{h \to 0^+} \frac{u(t_0 + h) - u(t_0)}{h}$$

The lower left Dini derivative is

$$D_- u(t_0) := \liminf_{h \to 0^-} \frac{u(t_0 + h) - u(t_0)}{h}$$

The upper right Dini derivative is

$$D^+ u(t_0) := \limsup_{h \to 0^+} \frac{u(t_0 + h) - u(t_0)}{h}$$

The upper left Dini derivative is

$$D^- u(t_0) := \limsup_{h \to 0^-} \frac{u(t_0 + h) - u(t_0)}{h}$$

Definition

A function $\alpha \in C([a,b])$ is a C^2 lower solution of the ODE if

1. for any $t_0 \in (a,b)$, either $D_-\alpha(t_0) < D_+\alpha(t_0)$, or there exist an open interval $I_0 \subset (a,b)$ with $t_0 \in I_0$ and a function $\alpha_0 \in C^1(I_0)$ such that
   - i. $\alpha(t_0) = \alpha_0(t_0)$ and $\alpha(t) \geq \alpha_0(t)$ for all $t \in I_0$
   - ii. $\alpha_0''(t_0)$ exists and $\alpha_0''(t_0) \geq f(t_0.\alpha_0(t_0), \alpha_0'(t_0))$
2. $a_1\alpha(a) - a_2 D_+\alpha(a) \leq A$, and $b_1\alpha(b) + b_2 D_-\alpha(b) \leq B$

Definition

A function $\beta \in C([a,b])$ is a C^2 upper solution of the ODE if

1. for any $t_0 \in (a,b)$, either $D_-\beta(t_0) > D_+\beta(t_0)$, or there exist an open interval $I_0 \subset (a,b)$ with $t_0 \in I_0$ and a function $\beta \in C^1(I_0)$ such that
   - i. $\beta(t_0) = \beta(t_0)$ and $\beta(t) \leq \beta_0(t)$ for all $t \in I_0$
   - ii. $\beta_0''(t_0)$ exists and $\beta_0''(t_0) \leq f(t_0.\beta_0(t_0), \beta_0'(t_0))$
2. $a_1\beta(a) - a_2 D_+\beta(a) \geq A$, and $b_1\beta(b) + b_2 D_-\beta(b) \geq B$

A Theorem

Theorem

Let $\alpha_i \in C([a,b])$ *be* C^2 *lower solutions of the ODE, for* $i = 1, \ldots, n$. *Then the function*

$$\alpha(t) = \max_{1 \leq i \leq n} \alpha_i(t), \; t \in [a,b]$$

is also a C^2 *lower solution of the ODE.*
Similarly, if $\beta_j \in C([a,b])$ *are* C^2 *upper solutions of the ODE, for* $j = 1, \ldots, m$. *Then the function*

$$\beta(t) = \min_{1 \leq j \leq m} \beta_j(t), \; t \in [a,b]$$

is also a C^2 *upper solution of the ODE*

Proof

Boundary condition left as exercise to reader.

Let $t_0 \in (a, b)$. For the first condition, we consider 2 cases:

1. Suppose there exists $g \in \{a_i\}_i$ such that the second part of the or statement is true. That is, there is g such that $g(t_0) = \alpha(t_0)$, and it satisfies that there is some interval $I_0 \subseteq (a, b)$ open, $t_0 \in I_0$, and some $\alpha_0 \in C^1(I_0)$ such that

   $$\alpha_0(t_0) = g(t_0)$$

   $$\alpha_0''(t_0) \geq f(t_0, \alpha_0(t_0), \alpha_0'(t_0))$$

   and for any $t \in I_0$,

   $$\alpha_0(t) \leq g(t)$$

   But wait...

   $$\alpha_0(t_0) = g(t_0) = \alpha(t_0)$$

   $$\alpha_0(t) \leq g(t) \leq \alpha(t)$$

   so we can use the same α_0.

Proof

Now suppose case 1 is not true. Then for all α_i such that $\alpha_i(t_0) = \alpha(t_0)$, $D_-\alpha_i(t_0) < D_+\alpha_i(t_0)$. Let $g \in \{\alpha_i\}_i$ such that $g(t_0) = \alpha(t_0)$. Note

$$D_+\alpha(t_0) = \liminf_{h\to 0^+} \frac{\alpha(t_0+h)-\alpha(t_0)}{h} \geq \liminf_{h\to 0^+} \frac{g(t_0+h)-g(t_0)}{h} = D_+g(t_0)$$

$$D_-\alpha(t_0) = \liminf_{h\to 0^-} \frac{\alpha(t_0+h)-\alpha(t_0)}{h} \leq \liminf_{h\to 0^-} \frac{g(t_0+h)-g(t_0)}{h} = D_-g(t_0)$$

so

$$D_-\alpha(t_0) \leq D_-g(t_0) < D_+g(t_0) \leq D_+\alpha(t_0)$$

the first part of the or statement is true.

Another theorem

Theorem

Let $A, B \in \mathbb{R}$, $a_1, b_1 \in \mathbb{R}$, $a_2, b_2 \in \mathbb{R}^+$. *Assume* $\alpha, \beta \in C([a,b])$ *are* C^2 *lower and upper solutions of the ODE such that* $\alpha \leq \beta$. *Let*

$$E := \{(t,u,v) \in [a,b] \times \mathbb{R}^2 \mid \alpha(t) \leq u \leq \beta(t)\}$$

and assume the function $f : E \to \mathbb{R}$ *is continuous and bounded. Then the ODE has at least 1 solution* $u \in C^2([a,b])$ *such that for all* $t \in [a,b]$,

$$\alpha(t) \leq u(t) \leq \beta(t)$$

Proof

We consider the modified problem

$$u'' - u = f(t - \gamma(t,u), u') - \gamma(t,u)$$
$$u(a) - a_2 u'(a) = A + (1 - a_1)\gamma(a, u(a))$$
$$u(b) + b_2 u'(b) = B + (1 - b_1)\gamma(b, u(b))$$

where

$$\gamma(t,u) = \max\{\alpha(t), \min\{u, \beta(t)\}\}$$

Claim 1: The modified problem has at least 1 solution. To prove this claim, write the modified problem as $Lu = Nu$, where

$$Lu = (u'' - u, u(a) - a_2 u'(a), u(b) + b_2 u'(b))$$

$$Nu = (f(t - \gamma(t,u), u') - \gamma(t,u), A + (1 - a_1)\gamma(a, u(a)), B + (1 - b_1)\gamma(b, u(b))$$

Claim that L is an invertible operator, so we can write $u = L^{-1}Nu$, and we can apply Schauder Fixed Point Theorem.

Schauder Fixed Point Theorem

Theorem (Schauder Fixed Point Theorem)

Let X be a Banach space. Let $K \subset X$ be a compact, convex set, and let $g : K \to K$ be a continuous function. Then g has a fixed point.

Theorem (Ascoli-Arzela / Arzela-Ascoli / Arzela / Ascoli)

Let $(X; d)$ be a separable metric space and let $\mathcal{F} \subseteq C_b(X)$ be a family of functions. Assume that $\mathcal{F}$ is bounded and equicontinuous at every point $x \in X$. Then every sequence $\{f_n\}_n$ in $\mathcal{F}$ has a subsequence $\{f_{n_j}\}_j$ that converges pointwise to a function $g \in C_b(X)$ and uniformly on every compact subset of X.

Corollary

Let $(X; d)$ be a compact metric space. Then $\mathcal{F} \subseteq C(X)$ is compact if and only if is closed, bounded, and uniformly equicontinuous.

Proof

First, let's show that L is an invertible operator. We know that L is linear, because each of the terms is linear. Also, we can find that it has a unique kernel, because then $Lu = (0, 0, 0)$ if $u'' = u, u(a) = a_2 u'(a), u(b) = -b_2 u'(b)$ and we know that $a_2, b_2 > 0$. However, the zero function satisfies that, the kernel is trivial. By similar reasoning, one can get that L is surjective.

Now, notice that L is continuous, and so is N in the space of functions. Finally, notice that $L^{-1}N$'s image set is actually equicontinuous, because we can explicitly find that:

$$L^{-1}N(u) = g(t) = C_1 e^t + C_2 e^{-t} + \frac{1}{2}(e^t \int e^{-t}(f - \gamma)dt - e^{-t} \int e^t (f - \gamma)dt)$$

and since all the constants are bounded, as $(f - \gamma), u'$ are both bounded over $[a, b]$, so therefore the image set of $L^{-1}N(u)$ is equicontinuous as the derivative of $L^{-1}N(u)$ is bounded over all u in the image set.

Proof

From above, we know that:

$$L^{-1}N(u) = g(t) = C_1e^t + C_2e^{-t} + \frac{1}{2}(e^t\int e^{-t}(f-\gamma)dt - e^{-t}\int e^t(f-\gamma)dt)$$

and since C_1, C_2 depend continuously on $\int e^t(f-\gamma), \int e^{-t}(f-\gamma)$ and all those functions are bounded, C_1, C_2 vary continuously along functions u, meaning that $L^{-1}N(u)$ is continuous.

Thus, if we let I to be the image set of $L^{-1}N$, then $L^{-1}N : I \to I$ has that I is a family of equicontinuous functions, which means by Ascoli-Arzela it's compact. Therefore, by Schauder fixed point theorem, since $L^{-1}N$ is continuous, there is a solution u such that $L^{-1}N(u) = u$, implying that there is a solution u that solves the equation.

Proof

Claim 2: The solutions u to the modified problem are such that for all $t \in [a, b]$,

$$\alpha(t) \leq u(t) \leq \beta(t)$$

To prove this claim, let us assume toward contradiction that $u - \alpha$ has a negative minimum at some point t_0. If $t_0 \in (a, b)$ we have $D^- \alpha(t_0) \leq D^+ \alpha(t_0)$. Using the definition of a lower solution, we obtain the contradiction

$$0 \leq u''(t_0) - \alpha''(t_0) = f(t_0, \alpha(t_0), \alpha_0'(t_0)) + u(t_0) - \alpha_0(t_0) - \alpha_0''(t_0) < 0$$

In the case $t_0 = a$, we have $u'(a) - D^+ \alpha(a) \geq 0$, and obtain the contradiction

$$0 = A + (1 - a_1)\gamma(a, u(a)) - u(a) + a_2 u'(a) > A - a_1 \alpha(a) + a_2 D^+ \alpha(a) \geq 0$$

A similar argument holds for $t_0 = b$. Similar reasoning also shows $u(t) \leq \beta(t)$.

Viscosity Solutions

Semicontinuous Functions

Definition

Given an interval $I \subseteq \mathbb{R}$, a function $u : I \to \mathbb{R}$ is upper semicontinuous if:

$$\limsup_{n\to\infty} u(t_n) \leq u(t)$$

for all sequences $t_n \to t \in I$. A function $u : I \to \mathbb{R}$ is lower semicontinuous if:

$$\liminf_{n\to\infty} u(t_n) \geq u(t)$$

for all sequences $t_n \to t \in I$. If a function is both upper and lower semicontinuous, then it is continuous, and the converse is true.

Some examples

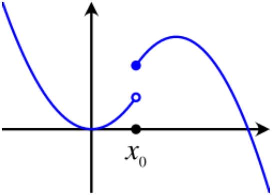


Figure: An upper semicontinuous function

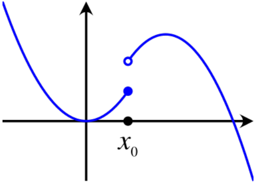


Figure: A lower semicontinuous function

Viscosity Sub and Super Solutions

Consider $F(t, u, u', u'') = \min\{u'' - f(t, u, u'), u - g\}$. and $g : [a, b] \to \mathbb{R}$ is of class C^2. Furthermore, we impose boundary conditions $u(a) = c, u(b) = d$.

Definition

An upper semicontinuous function $v : [a, b] \to \mathbb{R}$ is a viscosity subsolution if for every fixed $t_0 \in (a, b)$ that:

$$F(t_0, v(t_0), \phi'(t_0), \phi''(t_0)) \geq 0$$

for all $\phi \in C^2[a, b]$ such that $v - \phi$ has a local maximum at t_0 and $v(a) \leq u(a), v(b) \leq u(b)$. A lower semicontinuous function $w : [a, b] \to \mathbb{R}$ is a viscosity supersolution if for eveyr fixed $t_0 \in (a, b)$ that:

$$F(t_0, w(t_0), \phi'(t_0), \phi''(t_0)) \leq 0$$

for all $\phi \in C^2[a, b]$ such that $v - \phi$ has a local minimum at t_0. Furthermore, $w(a) \geq u(a), w(b) \geq u(b)$.

Degenerate Elliptic PDE properties

Suppose that v is a C^2 solution of $F = 0$. Then, we know that:

$$v - g \geq 0, v'' - f(u, v, v') \geq 0$$

with $v(a) = u(a), v(b) = u(b)$. Then, if $v - \phi$ has a local minimum at t_0 then we know that $\phi'(t_0) = v'(t_0)$, and that $\phi''(t_0) \leq v''(t_0)$. This means that:

$$F(t_0, v(t_0), \phi'(t_0), \phi''(t_0)) = \min\{v(t_0) - g(t_0), \phi''(t_0) - f(t_0, v, v')) \leq F(t_0, v, v', v'')$$

Therefore, v is actually a supersolution. Similar reasoning can show that v is also a subsolution.

Definition

A function v is a viscosity solution if and only if it is both a sub and super solution.

Bounds of Sub and Super Solutions

Theorem

In the case where $f(t,u,u') = 0$, if u_+, u_- are viscosity super and sub solutions respectively, then $u_+ \geq u_-$

By properties of semicontinuity, we can find that if ever $u_+ \leq u_-$, then they have to be equal at some point t_0. Then, from there by semicontinuity, we can bound u_+ from below by some continuous function, v_1 and we can bound u_- from above by some continuous function v_2 around t_0 such that $u_-(t_0) = v_2(t_0), u_+(t_0) = v_1(t_0)$. Suppose that we can construct $\phi_1, \phi_2 \in C^2[a,b]$ such that $\phi_1 \geq v_1, \phi_2 \leq v_2$ around t_0 and equality satisfied at t_0. Then, by the properties of sub and super solutions, we get that:

$$\phi_1'' \leq 0, \phi_2'' \geq 0$$

But then we know that $u_+ - \phi_1$ has a local maximum ato t_0 and $u_- - \phi_2$ has a local min, which means that $u_+ \geq u_-$ around t_0.

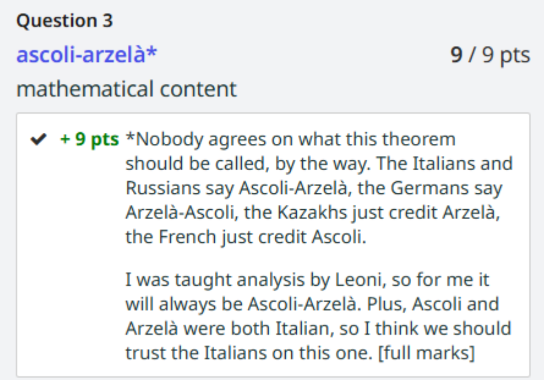
Bounds of Sub and Super Solutions

The second case is if we cannot construct such ϕ_1, ϕ_2. However, this would imply that v_1, v_2 are concave, convex, pointing away from each other, meaning that that $u_- \geq u_+$ in a neighborhood around t_0. However, since at the endpoints $u_- \leq u_+$, this is not possible.

Questions

Thanks

Uh idk how to make a good ending slide tbh. Thanks to everyone for showing up. Pretty Cool.



Question 3

ascoli-arzelà* **9** / 9 pts

mathematical content

✓ **+ 9 pts** *Nobody agrees on what this theorem should be called, by the way. The Italians and Russians say Ascoli-Arzelà, the Germans say Arzelà-Ascoli, the Kazakhs just credit Arzelà, the French just credit Ascoli.

I was taught analysis by Leoni, so for me it will always be Ascoli-Arzelà. Plus, Ascoli and Arzelà were both Italian, so I think we should trust the Italians on this one. [full marks]

Figure: Quote by Robert Trosten